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Weyl Ordering Symbol Method for Studying Wigner Function of the Damping Field

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Abstract The symbolic method (including normal ordering. antinormal ordering and Weyl ordering symbol) is usually utilized to tackle miscellaneous operators which have different commutative relations. Considering the Weyl ordering symbol's remarkable properties, we have efficiently and conveniently derived the Wigner distribution function for field damping in a squeezed bath and a vacuum bath respectively, and then examined the decoherence processes from the plots of Wigner function and its contour in quantum phase space. Alternatively, we can employ a general Wigner operator under phase space transform to calculate distribution function and discuss the damping process.

Keywords Weyl ordering · Wigner function · Similar transformation · Density operator · Decoherence

1 Introduction

Wigner distribution function (WDF) [1–5] is the most popularly used in studying quantum mechanics, quantum optics and quantum statistics, not only because it is an useful computational tool that enables one to transcribe operator equations into c-number equations, but also it has led to new concepts such as non-classic states of radiation field. As an important quantum phase-space technique, its two marginal distributions lead to measuring probability density in coordinate space and momentum space respectively. For a state, if Wigner function of the state has a negative part in the phase space, then the state is a nonclassical state. It gives the most analogous description of quantum mechanics in the phase space to classical statistical mechanics of Hamilton systems. Recently, direct measurement of quantum phase space distributions becomes possible by quantum state reconstruction [6].

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In dealling with Wigner distribution function, people always face to tackle miscellaneous operators which have different commutative relations. Putting operators in normal ordering [7–10], antinormal ordering [11] or Weyl ordering [12, 13] is usually needed. It has been well-known for long that the phase space formulation of quantum mechanics is based on Wigner's quasi-distribution function and the Weyl's correspondence [12] between operators and c-number phase-space functions. In our previous work [25], we have introduced a unitary squeezing operator $S=\exp[\frac{\psi}{2}(a^{\dagger 2}e^{-i\theta}-a^2e^{i\theta})]$ to simplify the density equation of field damping in a squeezed bath, i.e.

$$\frac{d}{dt}\rho=-\frac{\lambda}{2}(N+1)[a^{\dagger}a\rho-2a\rho a^{\dagger}+\rho a^{\dagger}a]-\frac{\lambda}{2}N[aa^{\dagger}\rho-2a^{\dagger}\rho a+\rho aa^{\dagger}]$$
$$+\frac{\lambda}{2}M[aa\rho-2a\rho a+\rho aa]+\frac{\lambda}{2}M^{*}[a^{\dagger}a^{\dagger}\rho-2a^{\dagger}\rho a^{\dagger}+\rho a^{\dagger}a^{\dagger}], \tag{1}$$

to be a vacuum bath case

$$\frac{d}{dt}\rho'=\frac{\lambda}{2}(2a\rho' a^{\dagger}-a^{\dagger}a\rho'-\rho' a^{\dagger}a),\quad \text{and}\quad \rho'=S^{-1}\rho S. \tag{2}$$

Utilizing the so-called entangled state representation method we have converted master equation into a c-number equation and derived the time evolution density of state

$$\rho(t)=S\rho'(t)S^{-1}=\mathcal{G}^2\sum_{i,\,j=1}^{2}\langle\alpha_j|\alpha_i\rangle^{(1-e^{-\lambda t})}S\left|\alpha_i e^{-\frac{\lambda}{2}t}\right\rangle\left\langle\alpha_j e^{-\frac{\lambda}{2}t}\right|S^{-1}. \tag{3}$$

In this letter, considering the Weyl ordering symbol's three remarkable properties: (1) the order of Bose operators within a Weyl ordered product can be permuted; (2) a Weyl ordered product can be integrated with respect to a C-number provided that the integration is convergent; (3) the Weyl ordering is invariant under the similar transformations [14, 15], we would like to employ the explicit Weyl-ordered form of Wigner operator and the technique of integration within Weyl ordered product (IWWOP) of operators [13] to derive the Wigner function characterizing the time evolution density of state, and then examine the decoherence processes of the damping field.

We arrange this letter as follows. In Sect. 2, we briefly review Wigner operator, Weyl ordering and their properties. In Sect. 3, by virtue of the Weyl ordering technique, we shall derive the Wigner function $W(q,p;t)=\mathrm{Tr}[\rho(t)\Delta(q,p)]$ for a damping field in a squeezed bath with initial state of the superposition of two coherent squeezed state, and show the decoherence process from the plots of Wigner function and its contour. In Sect. 4, considering a simplified damping field in a vacuum bath, we obtain the corresponding Wigner function and exhibit the decoherence process of this damping system. In Appendix, we find that a classical canonical transform $(q,p)\to(-\mathcal{C}q+\mathcal{A}p,\mathcal{D}q-\mathcal{B}p)$ generates a more general Wigner transform $\Delta(-\mathcal{C}q+\mathcal{A}p,\mathcal{D}q-\mathcal{B}p)$, which can be employed to derive the distribution function and discuss the damping process.

2 Preview of the Weyl Ordering and Its Properties

Weyl ordering is a useful one but quite troublesome because its definition is complicated. The Weyl ordering of operators is a recipe of quantizing a classical polynomial $q^m p^r$ of q

and p,

$$q^m p^r \to \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} Q^{m-l} P^r Q^l, \quad [Q, P] = i\hbar, \tag{4}$$

where Q, P are the coordinate and momentum operator respectively. The Weyl ordering is tightly related to the Weyl correspondence rule which is a recipe for quantizing a classical function defined in classical phase space as an operator. According to this rule, the quantum operators, F, P_i, and Q_i corresponding to the classical quantities f, p_i and q_i are given by,

$$F(P, Q) = \iint dq dp f(p, q) \Delta(q, p), \tag{5}$$

where $\Delta(q, p)$ denotes the Wigner operator [16–18]. By introdcuing the bosonic annihilation, creation operators

$$a = \frac{1}{\sqrt{2}}(Q + iP), \qquad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP), \tag{6}$$

and $\alpha = \frac{1}{\sqrt{2}}(q + ip)$ we can see

$$\Delta(p, q) \to \Delta(\alpha, \alpha^*) = \int \frac{d^2 z}{\pi^2} |\alpha + z\rangle\langle \alpha - z| e^{\alpha z^* - \alpha^* z} = \frac{1}{\pi} :\exp\left[-2(a^\dagger - \alpha^*)(a - \alpha)\right]:, \tag{7}$$

where $\langle z| = \langle 0| \exp[-\frac{1}{2}|z|^2 + z^* a]$ is he coherent state [19–24], $: :$ denotes the normal ordering symbol [7–10]. Note that the right hand side of (4) is in Weyl ordering, so we introduced in Ref. [13] the Weyl ordering symbol $\vdots\ \vdots$ and invented the technique of IWWOP, within $\vdots\ \vdots$ the Bose operators are permutable. As a result we found the Weyl ordering form of the Wigner operator is

$$\Delta(p, q) = \vdots\, \delta(q - Q)\delta(p - P) \,\vdots. \tag{8}$$

It then follows that the Weyl correspondence formula (5) can be recast to

$$\vdots\, f(P, Q) \,\vdots = \iint dq dp f(p, q) \Delta(p, q), \tag{9}$$

which means that the Weyl ordered operator $\vdots\, f(P, Q) \,\vdots$ is obtained by just replacing p, q in $f(p, q)$ by P, Q with the function form invariant, the exponential operator itself is in Weyl ordering. Thus using the IWWOP technique we have

$$\Delta(\alpha, \alpha^*) = \int \frac{d^2 z}{2\pi^2} \vdots \exp[z(a^\dagger - \alpha^*) - z^*(a - \alpha)] \vdots = \frac{1}{2} \vdots\, \delta(a^\dagger - \alpha^*)\delta(a - \alpha) \,\vdots. \tag{10}$$

Supposing an operator S engenders a similar transform

$$S a S^{-1} = \mu a + \nu a^\dagger, \qquad S a^\dagger S^{-1} = \sigma a + \tau a^\dagger, \tag{11}$$

where $\mu\tau - \sigma\nu = 1$ keeps

$$[\mu a + \nu a^\dagger, \sigma a + \tau a^\dagger] = 1, \tag{12}$$

we shall prove that S operator can “run across” the “border” of $\vdots\ \vdots$. In other words, the Weyl ordering is invariant under the similar transformations, which means if

$$F(a^{\dagger},a)=2\int d^{2}\alpha f(\alpha^{*},\alpha)\Delta(\alpha,\alpha^{*})=\vdots\, f(a^{\dagger},a)\,\vdots, \tag{13}$$

where $f(\alpha^{*},\alpha)$ is the classical Weyl correspondence of $F(a^{\dagger},a)$, then

$$SF(a^{\dagger},a)S^{-1}=F(\mu a+\nu a^{\dagger},\sigma a+\tau a^{\dagger})=\vdots\, f(\mu a+\nu a^{\dagger},\sigma a+\tau a^{\dagger})\,\vdots. \tag{14}$$

The proof is as follows. Using (7), (11) we have

$$\begin{aligned}S\Delta(\alpha,\alpha^{*})S^{-1}&=\int\frac{d^{2}z}{2\pi^{2}}:\exp\left\{-|z|^{2}\left(\sigma\nu+\frac{1}{2}\right)+z(\sigma a+\tau a^{\dagger}-\alpha^{*})\right.\\&\qquad\left.-z^{*}(\mu a+\nu a^{\dagger}-\alpha)+\frac{1}{2}(\sigma\tau z^{2}+\mu\nu z^{*2})\right\}:\\&=\frac{1}{\pi}:\exp\{-2(a^{\dagger}-\mu\alpha^{*}+\sigma\alpha)(a-\tau\alpha+\nu\alpha^{*})\}:.\end{aligned} \tag{15}$$

Performing S operator on (13) and using (7) and (10), (15) we have

$$\begin{aligned}SF(a^{\dagger},a)S^{-1}&=\frac{1}{\pi}2\int d^{2}\alpha f(\alpha^{*},\alpha):\exp\{-2(a^{\dagger}-\mu\alpha^{*}+\sigma\alpha)(a-\tau\alpha+\nu\alpha^{*})\}:\\&=\int d^{2}\alpha f(\mu\alpha+\nu\alpha^{*},\sigma\alpha+\tau\alpha^{*})\,\vdots\,\delta(a^{\dagger}-\alpha^{*})\delta(a-\alpha)\,\vdots\\&=\vdots\, f(\mu a+\nu a^{\dagger},\sigma a+\tau a^{\dagger})\,\vdots.\end{aligned} \tag{16}$$

Thus (14) is proved. Equation (16) is a very useful operator formula for recasting given operators into their Weyl ordering forms. Also we have demonstrated Dirac’s δ-operator function within Weyl ordering symbol $\vdots\ \vdots$, which brings convenience in studying some problems of quantum statistics and quantum optics. The advantage of introduction of Weyl ordering symbol $\vdots\ \vdots$ is manifestly shown in this work.

3 Wigner Function of Damping Field in a Squeezed Bath

In [25], when $t=0$, if the system is the superposition of two coherent squeezed states, i.e.

$$\rho(t=0)\equiv\rho_{0}=\mathcal{G}^{2}\sum_{i,j=1,2}S|\alpha_{i}\rangle\langle\alpha_{j}|S^{-1}, \tag{17}$$

where unitary operator $S=\exp[\frac{\psi}{2}(a^{\dagger2}e^{-i\theta}-a^{2}e^{i\theta})]$ engenders squeezing transformation

$$SaS^{-1}=a\cosh\psi-a^{\dagger}\sinh\psi e^{-i\theta},\qquad Sa^{\dagger}S^{-1}=a^{\dagger}\cosh\psi-a\sinh\psi e^{i\theta}, \tag{18}$$

we derived the time evolution density of state has shown as (3). From $W(q,p;t)=\mathrm{Tr}[\rho(t)\Delta(q,p)]$, we have

$$W(q,p;t=0)=\mathrm{Tr}[\rho_{0}\Delta(q,p)]=\mathcal{G}^{2}\,\mathrm{Tr}\left[\sum_{i,j=1,2}S|\alpha_{i}\rangle\langle\alpha_{j}|S^{-1}\Delta(q,p)\right], \tag{19}$$

in which existing of S makes it more difficulty to perform trace. Note that the Weyl ordering is invariant under the similar transformations (shown in Sect. 2), we shall convert $|\alpha_i\rangle\langle\alpha_j|$ to be its Weyl ordering. Utilizing Mehta formula [26]

$$\rho = 2\int \frac{d^2\beta}{\pi} \vdots \langle -\beta|\rho|\beta\rangle \exp[2(\beta^* a - \beta a^\dagger + a^\dagger a)] \vdots, \tag{20}$$

where $|\beta\rangle$ denotes the coherent state representation, we have

$$\begin{aligned} |\alpha_i\rangle\langle\alpha_j| &= 2\int \frac{d^2\beta}{\pi} \vdots \langle -\beta|.\alpha_i\rangle\langle\alpha_j.|\beta\rangle \exp[2(\beta^* a - \beta a^\dagger + a^\dagger a)] \vdots \\ &= 2 \vdots \exp\left[-\frac{|\alpha_i|^2 + |\alpha_j|^2}{2} - \alpha_j^*\alpha_i + 2\alpha_j^* a + 2\alpha_i a^\dagger - 2a^\dagger a\right] \vdots, \end{aligned} \tag{21}$$

where we have used the following integral formula

$$\begin{aligned} &\int \frac{d^2\alpha}{\pi} \exp[h|\alpha|^2 + s\alpha + \eta\alpha^* + f\alpha^2 + g\alpha^{*2}] \\ &\quad = \frac{1}{\sqrt{h^2 - 4fg}} \exp\left[\frac{-hs\eta + s^2 g + \eta^2 h}{h^2 - 4fg}\right], \end{aligned} \tag{22}$$

with convergent condition $\mathrm{Re}(h + f + g) < 0$ or $\mathrm{Re}(h - f - g) < 0$, where h, s, η, f and g are real parameters. Utilizing (14), (18) and (21) we can obtain

$$\begin{aligned} \rho_0 = 2\mathcal{G}^2 \sum_{i,j=1,2} \vdots \exp\Big[&-\frac{|\alpha_i|^2 + |\alpha_j|^2}{2} - \alpha_j^*\alpha_i + 2(\alpha_j^* \cosh\psi - \alpha_i e^{i\theta}\sinh\psi)a \\ &- 2(a^\dagger a\cosh^2\psi + a^\dagger a\sinh^2\psi - a^2 e^{i\theta}\sinh\psi\cosh\psi - a^{\dagger 2}e^{-i\theta}\sinh\psi\cosh\psi) \\ &+ 2(\alpha_i\cosh\psi - \alpha_j^* e^{-i\theta}\sinh\psi)a^\dagger\Big] \vdots \equiv \vdots F(a^\dagger, a) \vdots. \end{aligned} \tag{23}$$

On the other hand, based on Wigner operator in coherent state representation (shown in (7)), we shall convert ρ_0 in (23) to be its normal ordering, which can be obtained by replacing $(a,\ a^\dagger)$ in (23) by (α, α^*) and using Wigner operator on the second expression of (7), we have

$$\begin{aligned} \rho_0 &= 4\mathcal{G}^2 \sum_{i,j=1}^{2} : \exp\left[-\frac{|\alpha_i|^2 + |\alpha_j|^2}{2} - \alpha_j^*\alpha_i - 2a^\dagger a\right] \int \frac{d^2\alpha}{\pi} \\ &\quad \times \exp[2\alpha(\alpha_j^*\cosh\psi - \alpha_i e^{i\theta}\sinh\psi + a^\dagger) + 2\alpha^*(\alpha_i\cosh\psi - \alpha_j^*\sinh\psi e^{-i\theta} + a) \\ &\quad - 4\cosh^2\psi|\alpha|^2 + 2\alpha^2\sinh\psi\cosh\psi e^{i\theta} + 2\alpha^{*2}\sinh\psi\cosh\psi e^{-i\theta}] : \\ &= \frac{\mathcal{G}^2}{\cosh\psi} \sum_{i,j=1,2} : \exp\Big[-\frac{|\alpha_i|^2 + |\alpha_j|^2}{2} - \alpha_j^*\alpha_i - 2a^\dagger a \\ &\quad + AB + \frac{\sinh\psi}{2\cosh\psi}(e^{-i\theta}A^2 + e^{i\theta}B^2)\Big] :, \end{aligned} \tag{24}$$

where

$$A=\alpha_j^*\cosh\psi-\alpha_i e^{i\theta}\sinh\psi+a^\dagger,\qquad B=\alpha_i\cosh\psi-\alpha_j^*\sinh\psi e^{-i\theta}+a. \tag{25}$$

Substituting (24) into (9) and using Wigner operator on the first expression of (7), we derive the Wigner distribution function of initial state ρ_0

$$\begin{aligned}W(z,z^*;0)&=\mathrm{Tr}[\rho_0\Delta(z,z^*)]=\frac{1}{\pi}\mathrm{Tr}\left[\rho_0\int\frac{d^2\beta}{\pi}|z+\beta\rangle\langle z-\beta|e^{z\beta^*-z^*\beta}\right]\\&=\frac{\mathcal{G}^2}{\pi}\sum_{i,j=1,2}\exp\left[-\frac{|\alpha_i|^2+|\alpha_j|^2}{2}-\alpha_j^*\alpha_i-2(\sinh^2\psi+\cosh^2\psi)|z|^2\right.\\&\quad+2(\alpha_j^*\cosh\psi-\alpha_i e^{i\theta}\sinh\psi)z+2(\alpha_i\cosh\psi-\alpha_j^*e^{-i\theta}\sinh\psi)z^*\\&\quad\left.+2(z^2e^{i\theta}+z^{*2}e^{-i\theta})\sinh\psi\cosh\psi\right].\end{aligned}\tag{26}$$

If $t\neq 0$, similarly, we can obtain $\rho(t)$ within the Weyl ordering symbol,

$$\begin{aligned}\rho(t)&=\mathcal{G}^2\sum_{i,j=1}^{2}\langle\alpha_j|\alpha_i\rangle^{(1-e^{-\lambda t})}S\left|\alpha_i e^{-\frac{\lambda t}{2}}\right\rangle\!\left\langle\alpha_j e^{-\frac{\lambda t}{2}}\right|S^{-1}\\&=2\mathcal{G}^2\sum_{i,j=1}^{2}\langle\alpha_j|\alpha_i\rangle^{(1-e^{-\lambda t})}\,\vdots\,\exp\left[-e^{-\lambda t}\frac{|\alpha_i|^2+|\alpha_j|^2}{2}-\alpha_j^*\alpha_i e^{-\lambda t}\right.\\&\quad+2e^{-\frac{\lambda t}{2}}(\alpha_j^*\cosh\psi-\alpha_i\sinh\psi e^{i\theta})a+2e^{-\frac{\lambda t}{2}}(\alpha_i\cosh\psi-\alpha_j^*\sinh\psi e^{-i\theta})a^\dagger\\&\quad\left.-2a^\dagger a(\cosh^2\psi+\sinh^2\psi)+2(a^2e^{i\theta}+a^{\dagger 2}e^{-i\theta})\sinh\psi\cosh\psi\right]\vdots\,.\end{aligned}\tag{27}$$

It then follows

$$\begin{aligned}\rho(t)&=\frac{\mathcal{G}^2}{\cosh\psi}\sum_{i,j=1}^{2}\langle\alpha_j|\alpha_i\rangle^{(1-e^{-\lambda t})}:\exp\left[-e^{-\lambda t}\frac{|\alpha_i|^2+|\alpha_j|^2}{2}-\alpha_j^*\alpha_i e^{-\lambda t}-2a^\dagger a\right.\\&\quad\left.+AB+\frac{\sinh\psi}{2\cosh\psi}(A^2e^{-i\theta}+B^2e^{i\theta})\right]:,\end{aligned}\tag{28}$$

where

$$\begin{aligned}A&=\alpha_j^*e^{-\frac{\lambda t}{2}}\cosh\psi-\alpha_i e^{-\frac{\lambda t}{2}}e^{i\theta}\sinh\psi+a^\dagger,\\B&=\alpha_i e^{-\frac{\lambda t}{2}}\cosh\psi-\alpha_j^*e^{-\frac{\lambda t}{2}}e^{-i\theta}\sinh\psi+a.\end{aligned}\tag{29}$$

Thus the Wigner function of time evolution of state reads

$$\begin{aligned}W(z,z^*;t)&=\frac{\mathcal{G}^2}{\pi}\sum_{i,j=1,2}\langle\alpha_j|\alpha_i\rangle^{(1-e^{-\lambda t})}\exp\left[-e^{-\lambda t}\frac{|\alpha_i|^2+|\alpha_j|^2}{2}\right.\\&\quad-e^{-\lambda t}\alpha_j^*\alpha_i-2(2\sinh^2\psi+1)|z|^2\end{aligned}$$

$$+ 2(\alpha_j^* \cosh\psi - \alpha_i e^{i\theta}\sinh\psi)z + 2(\alpha_i\cosh\psi - \alpha_j^* e^{-i\theta}\sinh\psi)z^* \\ + 2(z^2 e^{i\theta} + z^{*2}e^{-i\theta})\sinh\psi\cosh\psi\Big]. \tag{30}$$

In Appendix, we show that the unitary operator S, which maps into a classical canonical transform $(q, p) \to (-\mathcal{C}q + \mathcal{A}p, \mathcal{D}q - \mathcal{B}p)$, generates a more general Wigner transform $\Delta(p, q) \to \Delta(-\mathcal{C}q + \mathcal{A}p, \mathcal{D}q - \mathcal{B}p)$ with the constraint $\mathcal{AD} - \mathcal{BC} = 1$. Thus we can employ this general Wigner operator under phase space transform to calculate distribution function and discuss the damping process.

In order to further view the decoherence of this damping system, we discuss changes in Wigner function and its contour plot with varying time in Figs. 1 and 2. In Fig. 1 we can clearly see that at initial time two peaks are compressed and deflexion, which can be well interpreted from the squeezing parameter ψ and rotation parameter θ in unitary squeezing operator S. The negative part means that the initial state is a nonclassical state. With time increasing, we can see the peaks values increase and the two peaks tend to combination,

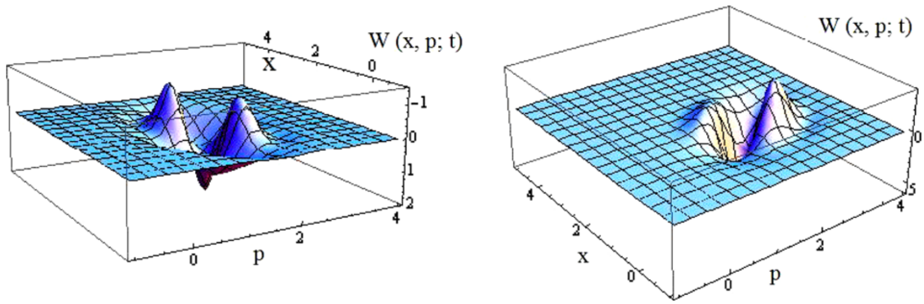

Fig. 1 Wigner function of $\rho(t)$ at $\frac{\mathcal{G}^2}{\pi} = 1$, $\alpha_1 = 1 + 2i$, $\alpha_2 = 3 + 4i$, $z = x + ip$, $\psi = \frac{1}{2}$, $\theta = \frac{\pi}{4}$ and different time: (1) $t = 0$ and (2) $t = \frac{\ln 2}{\lambda}$

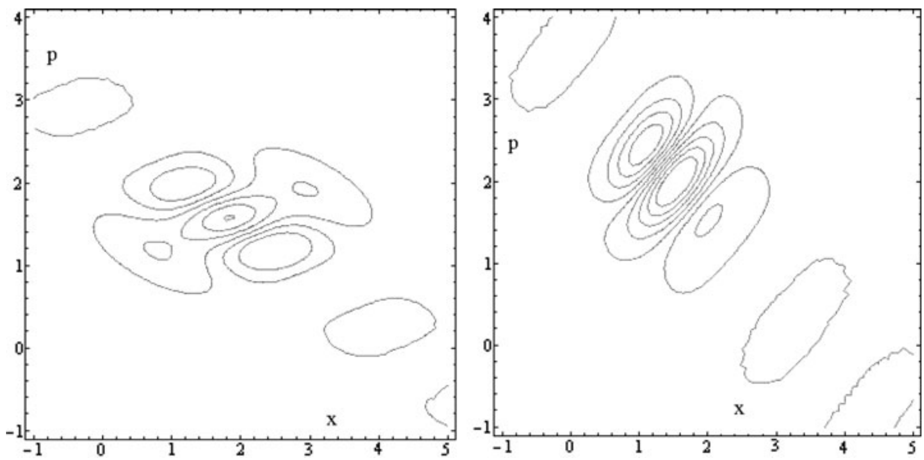

Fig. 2 Contour plots of Wigner function of $\rho(t)$ at $t = 0$ and $t = \frac{\ln 2}{\lambda}$

which indicates the decoherence. From the contour plots (shown in Fig. 2) of the Wigner function we can more clearly see this point.

4 Wigner Function of Damping Field in a Vacuum Bath

Employing the unitary transformation $S=\exp[\frac{\psi}{2}(a^{\dagger 2}e^{-i\theta}-a^2e^{i\theta})]$, we have transformed the damping field in a squeezed bath into a simple case (shown in (2)), which describes a field damping in a vacuum bath. If the initial state is the superposition of two coherent states, i.e.

$$\rho_0'=\mathcal{G}^2\sum_{i,j=1,2}|\alpha_i\rangle\langle\alpha_j|, \tag{31}$$

its time evolution of density operator reads

$$\rho'(t)=\mathcal{G}^2\sum_{i,j=1}^{2}\langle\alpha_j|\alpha_i\rangle^{(1-e^{-\lambda t})}\left|\alpha_i e^{-\frac{\lambda}{2}t}\right\rangle\left\langle\alpha_j e^{-\frac{\lambda}{2}t}\right|, \tag{32}$$

from which we can see

$$\begin{aligned}W(z,z^*;t)=\frac{\mathcal{G}^2}{\pi}\sum_{i,j=1,2}\langle\alpha_j|\alpha_i\rangle^{(1-e^{-\lambda t})}\exp\bigg[-2|z|^2-e^{-\lambda t}\frac{|\alpha_i|^2+|\alpha_j|^2}{2}\\ +2e^{-\frac{\lambda t}{2}}(\alpha_i z^*+\alpha_j^* z)-e^{-\lambda t}\alpha_i\alpha_j^*\bigg],\end{aligned} \tag{33}$$

and when $t=0$

$$W(z,z^*;0)=\frac{\mathcal{G}^2}{\pi}\sum_{i,j=1,2}\exp\left[-2|z|^2-\frac{|\alpha_i|^2+|\alpha_j|^2}{2}+2\alpha_i z^*+2\alpha_j^* z-\alpha_i\alpha_j^*\right]. \tag{34}$$

If $t\to\infty$, from (32) we have $\rho'(t)\Longrightarrow\rho'(\infty)=\mathcal{G}^2\sum_{i,j=1}^{2}\langle\alpha_j|\alpha_i\rangle|0\rangle\langle 0|$, which demonstrates that if time is long enough the time evolution of state degenerates to be a vacuum state. Corresponding Wigner function is

$$W'(z,z^*;\infty)=\frac{\mathcal{G}^2}{\pi}\sum_{i,j=1,2}\langle\alpha_j|\alpha_i\rangle\exp[-2|z|^2] \tag{35}$$

which is just (33) at $t=\infty$.

If initial states is the superposition of two coherent, plots of the Wigner function and its contour have been shown in Fig. 3 with $\alpha_1=1+2i$, $\alpha_2=2+i$ for different time. When $t=0$, double peaks show the well-known Winger function characteristic of the superposition of two coherent states. With time increasing, we also can see one peak is gradually close to the other. To our knowledge, decoherence shorten the distance between peaks. If time increase enough, from (35) we can see the system will degenerate into a vacuum state, whose Wigner function and its contour have been shown in Fig. 4.

In summary, by virtue of Weyl ordering symbol and its well properties, we can deal with the miscellaneous operators which have different commutative relations. Employing Weyl ordering symbol method, we have calculated the Wigner functions for field damping in a

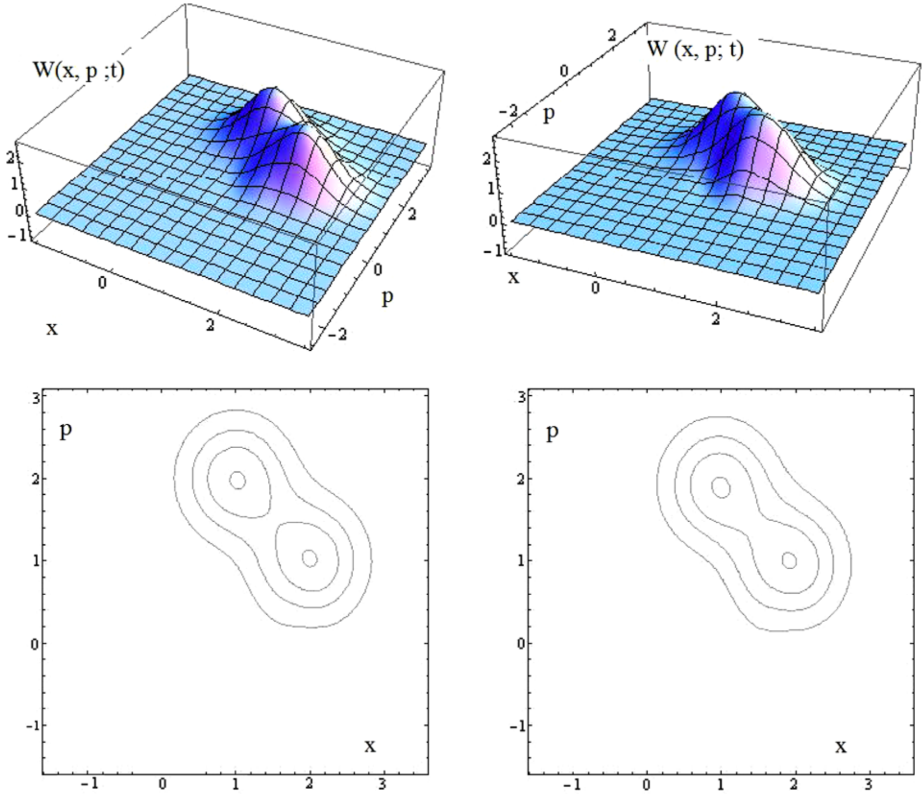

Fig. 3 Wigner function and its contour plot of $\rho'(t)$ at $\frac{\mathcal{G}^2}{\pi} = 1$, $\alpha_1 = 1 + 2i$, $\alpha_2 = 2 + i$, $z = x + ip$ and different time: (1) $t = 0$ and (2) $t = \frac{1}{8}$

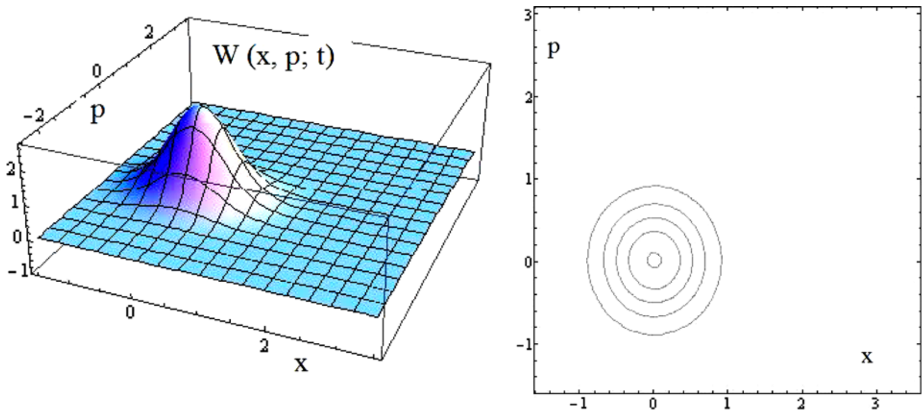

Fig. 4 Wigner function and its contour plot of the vacuum state

squeezed bath and vacuum bath, respectively, and then analyzed the decoherence processes from quantum phase-space distribution plots. The results show that this approach actually is efficient and convenient for studying quantum phase-space distribution.

Appendix

From (11), we can see unitary operator S engenders a similar transform

$$\begin{aligned} S^{-1}QS &= \frac{1}{2}[(\tau+\mu-\sigma-\nu)Q+(\tau-\mu-\sigma+\nu)iP] \equiv \mathcal{A}Q+\mathcal{B}P, \\ S^{-1}PS &= \frac{1}{2i}[(\tau-\mu+\sigma-\nu)Q+(\tau+\mu+\sigma+\nu)iP] \equiv \mathcal{C}Q+\mathcal{D}P, \end{aligned} \tag{36}$$

i.e.

$$S^{-1}\begin{pmatrix} Q \\ P \end{pmatrix} S = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}, \quad \mathcal{A}\mathcal{D}-\mathcal{B}\mathcal{C}=1. \tag{37}$$

Utilizing (8) and (37), we have

$$\begin{aligned} S^{-1}\Delta(p,q)S &= S^{-1} \vdots\, \delta(q-Q)\delta(p-P) \,\vdots\, S = \vdots\, \delta(q-\mathcal{A}Q-\mathcal{B}P)\delta(p-\mathcal{C}Q-\mathcal{D}P) \,\vdots \\ &= \vdots\, \delta(-\mathcal{C}q+\mathcal{A}p-P)\delta(\mathcal{D}q-\mathcal{B}p-Q) \,\vdots \\ &= \Delta(-\mathcal{C}q+\mathcal{A}p, \mathcal{D}q-\mathcal{B}p), \end{aligned} \tag{38}$$

which is a more general Wigner transform mapping from the classical canonical transform $(q,p) \to (\mathcal{D}q-\mathcal{B}p, -\mathcal{C}q+\mathcal{A}p)$ in phase space. Considering (9) and (38), we have

$$\begin{aligned} &\Delta(-\mathcal{C}q+\mathcal{A}p, \mathcal{D}q-\mathcal{B}p) \\ &\quad = \int dp'dq'\delta(-\mathcal{C}q+\mathcal{A}p-p')\delta(\mathcal{D}q-\mathcal{B}p-q')\Delta(p',q') \\ &\quad = \frac{1}{\pi} : \int dp'dq'\delta(-\mathcal{C}q+\mathcal{A}p-p')\delta(\mathcal{D}q-\mathcal{B}p-q')\exp[-(q'-Q)^2-(p'-P)^2]: \\ &\quad = \frac{1}{\pi} : \exp[-(\mathcal{D}q-\mathcal{B}p-Q)^2-(-\mathcal{C}q+\mathcal{A}p-P)^2] :, \end{aligned} \tag{39}$$

and then

$$\begin{aligned} &\Delta(-\mathcal{C}q+\mathcal{A}p, \mathcal{D}q-\mathcal{B}p) \\ &\quad \Rightarrow \quad \Delta'(z,z^*) \\ &\qquad = \frac{1}{\pi} : \exp\left[-\left(\frac{\mathcal{D}+i\mathcal{B}}{\sqrt{2}}z+\frac{\mathcal{D}-i\mathcal{B}}{\sqrt{2}}z^*-\frac{a+a^\dagger}{\sqrt{2}}\right)^2\right. \\ &\qquad\quad \left. -\left(-\frac{\mathcal{C}+i\mathcal{A}}{\sqrt{2}}z-\frac{\mathcal{C}-i\mathcal{A}}{\sqrt{2}}z^*-\frac{a-a^\dagger}{\sqrt{2}i}\right)^2\right] : . \end{aligned} \tag{40}$$

Substituting (40) into (19)

$$\begin{aligned} W(z,z;0) &= \mathcal{G}^2\,\mathrm{Tr}\Big[\sum_{i,j=1,2}|\alpha_i\rangle\langle\alpha_j|S^{-1}\Delta(q,p)S\Big] = \mathcal{G}^2\,\mathrm{Tr}\Big[\sum_{i,j=1,2}|\alpha_i\rangle\langle\alpha_j|\Delta'(z,z^*)\Big] \\ &= \frac{\mathcal{G}^2}{\pi}\sum_{i,j=1,2}\exp\Big[-\frac{|\alpha_i|^2+|\alpha_j|^2}{2}+\alpha_j^*\alpha_i-\Big(\frac{\mathcal{D}+i\mathcal{B}}{\sqrt{2}}z+\frac{\mathcal{D}-i\mathcal{B}}{\sqrt{2}}z^*-\frac{\alpha_i+\alpha_j^*}{\sqrt{2}}\Big)^2 \\ &\quad -\Big(-\frac{\mathcal{C}+i\mathcal{A}}{\sqrt{2}}z-\frac{\mathcal{C}-i\mathcal{A}}{\sqrt{2}}z^*-\frac{\alpha_i-\alpha_j^*}{\sqrt{2}i}\Big)^2\Big], \end{aligned} \tag{41}$$

and then using (18) and (36), we identity

$$\begin{aligned} &\mathcal{A}=\cosh\varphi+\sinh\varphi\cos\theta, \qquad \mathcal{B}=\mathcal{C}=-\sinh\varphi\sin\theta, \\ &\mathcal{D}=\cosh\varphi-\sinh\varphi\cos\theta, \end{aligned} \tag{42}$$

then it follows

$$\begin{aligned} W(z,z;0) = \frac{\mathcal{G}^2}{\pi}\sum_{i,j=1,2}\exp\Big[&-\frac{|\alpha_i|^2+|\alpha_j|^2}{2}-\alpha_j^*\alpha_i-2(\sinh^2\psi+\cosh^2\psi)|z|^2 \\ &+2(\alpha_j^*\cosh\psi-\alpha_i e^{i\theta}\sinh\psi)z+2(\alpha_i\cosh\psi-\alpha_j^*e^{-i\theta}\sinh\psi)z^* \\ &+2(z^2e^{i\theta}+z^{*2}e^{-i\theta})\sinh\psi\cosh\psi\Big], \end{aligned} \tag{43}$$

which is just (26). In the same way, we can derive the Wigner function of time evolution $W(z,z;t)$.

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